

# Stochastic differential games with inside information

Olfa Draouil<sup>1</sup> and Bernt Øksendal<sup>2,3</sup>

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## Abstract

We study stochastic differential games of jump diffusions, where the players have access to inside information. Our approach is based on anticipative stochastic calculus, white noise, Hida-Malliavin calculus, forward integrals and the Donsker delta functional. We obtain a characterization of Nash equilibria of such games in terms of the corresponding Hamiltonians. This is used to study applications to insider games in finance, specifically optimal insider consumption and optimal insider portfolio under model uncertainty.

## 1 Introduction

In this paper we present a general method for solving *optimal insider games*, i.e. optimal insider games problems where the players has access to some future information about the system. This inside information in the control processes puts the problem outside the context of semimartingale theory, and we therefore apply general *anticipating white noise calculus*, including *forward integrals* and *Hida-Malliavin calculus*. Combining this with the *Donsker delta functional* for the random variable  $Y = (Y_1, Y_2)$  which represents the inside information, we are able to prove both a sufficient and a necessary maximum principle for the optimal insider games of such systems.

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<sup>1</sup>Department of Mathematics, University of Tunis El Manar, Tunis, Tunisia. Email: olfadraouil@hotmail.fr

<sup>2</sup>Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N-0316 Oslo, Norway. Email: oksendal@math.uio.no

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We now explain this in more detail:

The system we consider, is described by a stochastic differential equation driven by a Brownian motion  $B(t)$  and an independent compensated Poisson random measure  $\tilde{N}(dt, d\zeta)$ , jointly defined on a filtered probability space  $(\Omega, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$  satisfying the usual conditions. We assume that the inside information is of *initial enlargement* type. Specifically, we assume that the two inside filtrations  $\mathbb{H}^1, \mathbb{H}^2$  representing the information flows available to player 1 and player 2, respectively, have the form

$$\mathbb{H}^i = \{\mathcal{H}_t^i\}_{t \geq 0}, \text{ where } \mathcal{H}_t^i = \mathcal{F}_t \vee Y_i, \quad i = 1, 2 \quad (1.1) \quad \{\text{eq1.1}\}$$

for all  $t$ , where  $Y_i$  is a given  $\mathcal{F}_{T_0}$ -measurable random variable, for some fixed  $T_0 > T > t$ . Here the insider control process  $u(t) = (u_1(t), u_2(t))$ , where  $u_i(t)$  is the control of player  $i$ ;  $i=1,2$ . Thus we assume that the value at time  $t$  of our insider control process  $u_i(t)$  is allowed to depend on both  $Y_i$  and  $\mathcal{F}_t$ ;  $i = 1, 2$ . In other words,  $u_i$  is assumed to be  $\mathbb{H}^i$ -adapted for  $i = 1, 2$ . Therefore they have the form

$$u_i(t, \omega) = \bar{u}_i(t, Y_i, \omega) \quad (1.2) \quad \{\text{eq1.2}\}$$

for some function  $\bar{u}_i : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  such that  $\bar{u}_i(t, y_i)$  is  $\mathbb{F}$ -adapted for each  $y_i \in \mathbb{R}$ . For simplicity (albeit with some abuse of notation) we will in the following write  $u_i$  in stead of  $\bar{u}_i$ ;  $i = 1, 2$ . Consider a controlled stochastic process  $X(t) = X^u(t)$  of the form

$$\begin{cases} dX(t) = dX^u(t) = b(t, X(t), u_1(t), u_2(t), Y_1, Y_2)dt + \sigma(t, X(t), u_1(t), u_2(t), Y_1, Y_2)dB(t) \\ + \int_{\mathbb{R}} \gamma(t, X(t), u_1(t), u_2(t), Y_1, Y_2, \zeta) \tilde{N}(dt, d\zeta); \quad t \geq 0 \\ X(0) = x(Y) \in \mathbb{R}, \end{cases} \quad (1.3) \quad \{\text{eq2.1}\}$$

where  $u_i(t) = u_i(t, y_i)_{y_i=Y_i}$  is the control process of insider  $i$ ;  $i = 1, 2$ , and the (anticipating) stochastic integrals are interpreted as *forward integrals*, as introduced in [RV] (Brownian motion case) and in [DMØP1] (Poisson random measure case). A motivation for using forward integrals in the modelling of insider control is given in [BØ]. Let  $\mathcal{A}_i$  denote a given set of admissible  $\mathbb{H}^i$ -adapted controls  $u_i$  of player  $i$ , with values in  $\mathbf{A}_i \subset \mathbb{R}^d$ ,  $d \geq 1$ ;  $i = 1, 2$ . Denote  $\mathbb{U} = \mathbf{A}_1 \times \mathbf{A}_2$ . Then  $X(t)$  is  $\mathbb{F} \vee Y_1 \vee Y_2$ -adapted. The *performance functional*  $J_i(u)$ ;  $u = (u_1, u_2)$  of player  $i$  is defined by, writing  $y = (y_1, y_2)$  and  $dy = dy_1 dy_2$ ,

$$\begin{aligned} J_i(u) &= \mathbb{E} \left[ \int_0^T f_i(t, X(t), u_1(t), u_2(t), Y) dt + g_i(X(T), Y) \right] \\ &= \mathbb{E} \left[ \int_{\mathbb{R}^2} \left\{ \int_0^T f_i(t, x(t, y_1, y_2), u_1(t, y_1), u_2(t, y_2), y) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_t] dt \right. \right. \\ &\quad \left. \left. + g_i(x(T, y_1, y_2), y_1, y_2) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_T] \right\} dy \right]; \quad i = 1, 2. \end{aligned} \quad (1.4)$$

A *Nash equilibrium* for the game (3.3)-(3.8) is a pair  $\hat{u} = (\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$  such that

$$\sup_{u_1 \in \mathcal{A}_1} J_1(u_1, \hat{u}_2) \leq J_1(\hat{u}_1, \hat{u}_2) \quad (1.5) \quad \{\text{eq2.10}\}$$

and

$$\sup_{u_2 \in \mathcal{A}_2} J_2(\hat{u}_1, u_2) \leq J_2(\hat{u}_1, \hat{u}_2). \quad (1.6) \quad \{\text{eq2.11}\}$$

We use the Donsker delta functional of  $Y = (Y_1, Y_2)$  to find a Nash equilibrium for the game (3.3)-(3.8).

Here is an outline of the content of the paper:

- In Section 2 we define the Donsker delta functional.
- In Section 3 we present the general insider optimal control problem for the stochastic differential games.
- In Sections 4 and 5 we present a sufficient and a necessary maximum principle, respectively, for the insider game problem.
- In Section 6 we present the zero-sum game case where we distinguish two situations: Situation 1: Both players are still maximizing their own performance functional and situation 2: One of the players is maximizing and the other is minimizing the performance functional and we write the sufficient and necessary maximum principle corresponding for each situation.
- Then in Section 7 we illustrate our results by applying them to optimal insider consumption and optimal insider portfolio under model uncertainty.

## 2 The Donsker delta functional

**Definition 2.1** *Let  $(Y_1, Y_2) : \Omega^2 \rightarrow \mathbb{R}$  be a pair of random variables which also belongs to  $(\mathcal{S})^{*2}$ . Then a continuous functional*

$$\delta_{Y_1, Y_2}(\cdot) : \mathbb{R} \times \mathbb{R} \rightarrow (\mathcal{S})^* \quad (2.1) \quad \{\text{donsker}\}$$

*is called a Donsker delta functional of  $(Y_1, Y_2)$  if it has the property that*

$$\int_{\mathbb{R}^2} g(y_1, y_2) \delta_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2 = g(Y_1, Y_2) \quad a.s. \quad (2.2) \quad \{\text{donsker pr}\}$$

*for all (measurable)  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that the integral converges.*

For more information about the Donsker delta function and some explicit formulas for it, see [DrØ].

### 3 The general insider optimal control problem for the stochastic differential games

In this section, we formulate and prove a sufficient and a necessary maximum principle for general stochastic differential games (not necessarily zero-sum games) for insiders. The system we consider, is described by a stochastic differential equation driven by a Brownian motion  $B(t)$  and an independent compensated Poisson random measure  $\tilde{N}(dt, d\zeta)$ , jointly defined on a filtered probability space  $(\Omega, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$  satisfying the usual conditions. We assume that the inside information is of *initial enlargement* type. Specifically, we assume that the two inside filtrations  $\mathbb{H}^1, \mathbb{H}^2$  representing the information flows available to player 1 and player 2, respectively, have the form

$$\mathbb{H}^i = \{\mathcal{H}_t^i\}_{t \geq 0}, \text{ where } \mathcal{H}_t^i = \mathcal{F}_t \vee Y_i, \quad i = 1, 2 \quad (3.1) \quad \{\text{eq1.1}\}$$

for all  $t$ , where  $Y_i$  is a given  $\mathcal{F}_T$ -measurable random variable, for some fixed  $T > t$ . Here the insider control process  $u(t) = (u_1(t), u_2(t))$ , where  $u_i(t)$  is the control of player  $i$ ;  $i=1,2$ . Thus we assume that the value at time  $t$  of our insider control process  $u_i(t)$  is allowed to depend on both  $Y_i$  and  $\mathcal{F}_t$ ;  $i = 1, 2$ . In other words,  $u_i$  is assumed to be  $\mathbb{H}^i$ -adapted for  $i = 1, 2$ . Therefore they have the form

$$u_i(t, \omega) = \bar{u}_i(t, Y_i, \omega) \quad (3.2) \quad \{\text{eq1.2}\}$$

for some function  $\bar{u}_i : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  such that  $\bar{u}_i(t, y_i)$  is  $\mathbb{F}$ -adapted for each  $y_i \in \mathbb{R}$ . For simplicity (albeit with some abuse of notation) we will in the following write  $u_i$  in stead of  $\bar{u}_i$ ;  $i = 1, 2$ .

Consider a controlled stochastic process  $X(t) = X^u(t)$  of the form

$$\begin{cases} dX(t) = dX^u(t) = b(t, X(t), u_1(t), u_2(t), Y_1, Y_2)dt + \sigma(t, X(t), u_1(t), u_2(t), Y_1, Y_2)dB(t) \\ + \int_{\mathbb{R}} \gamma(t, X(t), u_1(t), u_2(t), Y_1, Y_2, \zeta) \tilde{N}(dt, d\zeta); \quad t \geq 0 \\ X(0) = x, \quad x \in \mathbb{R}, \end{cases} \quad (3.3) \quad \{\text{eq2.1}\}$$

where  $u_i(t) = u_i(t, y_i)_{y_i=Y_i}$  is the control process of insider  $i$ ;  $i = 1, 2$ , and the (anticipating) stochastic integrals are interpreted as *forward integrals*, as introduced in [RV] (Brownian motion case) and in [DMØP1] (Poisson random measure case). A motivation for using forward integrals in the modelling of insider control is given in [BØ]. Let  $\mathcal{A}_i$  denote a given set of admissible  $\mathbb{H}^i$ -adapted controls  $u_i$  of player  $i$ , with values in  $\mathbf{A}_i \subset \mathbb{R}^d$ ,  $d \geq 1$ ;  $i = 1, 2$ . Denote  $\mathbb{U} = \mathbf{A}_1 \times \mathbf{A}_2$ . Then  $X(t)$  is  $\mathbb{F} \vee Y_1 \vee Y_2$ -adapted, and hence using the definition of the Donsker delta functional  $\delta_{Y_1, Y_2}(y_1, y_2)$  of  $(Y_1, Y_2)$  we get

$$X(t) = x(t, Y_1, Y_2) = x(t, y_1, y_2)_{y_1=Y_1, y_2=Y_2} = \int_{\mathbb{R}^2} x(t, y_1, y_2) \delta_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2 \quad (3.4) \quad \{\text{eq6}\}$$

for some  $y_1, y_2$ -parametrized process  $x(t, y_1, y_2)$  which is  $\mathbb{F}$ -adapted for each  $y_1, y_2$ . Then, again by the definition of the Donsker delta functional and the properties of forward inte-

gration ([DrØ]), we can write

$$\begin{aligned}
X(t) &= x + \int_0^t b(s, X(s), u_1(s), u_2(s), Y_1, Y_2) ds + \int_0^t \sigma(s, X(s), u_1(s), u_2(s), Y_1, Y_2) dB(s) \\
&+ \int_0^t \int_{\mathbb{R}} \gamma(s, X(s), u_1(s), u_2(s), Y_1, Y_2, \zeta) \tilde{N}(ds, d\zeta) \\
&= x + \int_0^t b(s, x(s, Y_1, Y_2), u_1(s, Y_1), u_2(s, Y_2), Y_1, Y_2) ds \\
&+ \int_0^t \sigma(s, x(s, Y_1, Y_2), u_1(s, Y_1), u_2(s, Y_2), Y_1, Y_2) dB(s) \\
&+ \int_0^t \int_{\mathbb{R}} \gamma(s, x(s, Y_1, Y_2), u_1(s, Y_1), u_2(s, Y_2), Y_1, Y_2, \zeta) \tilde{N}(ds, d\zeta) \\
&= x + \int_0^t b(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2)_{y_1=Y_1, y_2=Y_2} ds \\
&+ \int_0^t \sigma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2)_{y_1=Y_1, y_2=Y_2} dB(s) \\
&+ \int_0^t \int_{\mathbb{R}} \gamma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2, \zeta)_{y_1=Y_1, y_2=Y_2} \tilde{N}(ds, d\zeta) \\
&= x + \int_0^t \int_{\mathbb{R}^2} b(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2) \delta_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2 ds \\
&+ \int_0^t \int_{\mathbb{R}^2} \sigma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2) \delta_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2 dB(s) \\
&+ \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}} \gamma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2, \zeta) \delta_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2 \tilde{N}(ds, d\zeta) \\
&= x + \int_{\mathbb{R}^2} \left[ \int_0^t b(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2) ds \right. \\
&+ \int_0^t \sigma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2) dB(s) \\
&+ \left. \int_0^t \int_{\mathbb{R}} \gamma(s, x(s, y_1, y_2), u_1(s, y_1), u_2(s, y_2), y_1, y_2, \zeta) \tilde{N}(ds, d\zeta) \right] \delta_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2
\end{aligned} \tag{3.5} \quad \{\text{eq7}\}$$

Comparing (3.4) and (3.5) we see that (3.4) holds if we choose  $x(t, y)$  for each  $y = (y_1, y_2)$  as the solution of the classical SDE

$$\begin{aligned}
dx(t, y_1, y_2) &= b(t, x(t, y_1, y_2), u_1(t, y_1), u_2(t, y_2), y_1, y_2) dt \\
&+ \sigma(t, x(t, y_1, y_2), u_1(t, y_1), u_2(t, y_2), y_1, y_2) dB(t) \\
&+ \int_{\mathbb{R}} \gamma(t, x(t, y_1, y_2), u_1(t, y_1), u_2(t, y_2), y_1, y_2, \zeta) \tilde{N}(dt, d\zeta); \quad t \geq 0
\end{aligned} \tag{3.6} \quad \{\text{eq8}\}$$

$$x(0, y) = x, \quad x \in \mathbb{R}, \tag{3.7}$$

The *performance functional*  $J_i(u)$ ;  $u = (u_1, u_2)$  of player  $i$  is defined by

$$\begin{aligned} J_i(u) &= \mathbb{E}\left[\int_0^T f_i(t, X(t), u_1(t), u_2(t))dt + g_i(X(T))\right] \\ &= \mathbb{E}\left[\int_0^T \left\{ \int_{\mathbb{R}^2} f_i(t, x(t, y_1, y_2), u_1(t, y_1), u_2(t, y_2), y_1, y_2) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_t] dt \right. \right. \\ &\quad \left. \left. + g_i(x(T, y_1, y_2), y_1, y_2) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_T] \right\} dy\right]; \quad i = 1, 2. \end{aligned} \quad (3.8)$$

A *Nash equilibrium* for the game (3.3)-(3.8) is a pair  $\hat{u} = (\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$  such that

$$\sup_{u_1 \in \mathcal{A}_1} J_1(u_1, \hat{u}_2) \leq J_1(\hat{u}_1, \hat{u}_2) \quad (3.9) \quad \{\text{eq2.10}\}$$

and

$$\sup_{u_2 \in \mathcal{A}_2} J_2(\hat{u}_1, u_2) \leq J_2(\hat{u}_1, \hat{u}_2). \quad (3.10) \quad \{\text{eq2.11}\}$$

## 4 A sufficient maximum principle

The problem (3.9)-(3.10) is a stochastic differential game with a standard (albeit parametrized) stochastic differential equation (3.6) for the state process  $x(t, y_1, y_2)$ , but with a non-standard performance functional given by (3.8). We can solve this problem by a modified maximum principle approach, as follows:

Define the *Hamiltonians*  $H_i : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \Omega \rightarrow \mathbb{R}$  by

$$\begin{aligned} H_i(t, x, y_1, y_2, u_1, u_2, p, q, r) &= H(t, x, y_1, y_2, u_1, u_2, p, q, r, \omega) \\ &= \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_t] f_i(t, x, u_1, u_2, y_1, y_2) + b(t, x, u_1, u_2, y_1, y_2) p \\ &\quad + \sigma(t, x, u_1, u_2, y_1, y_2) q + \int_{\mathbb{R}} \gamma(t, x, u_1, u_2, y_1, y_2) r(t, \zeta) \nu(d\zeta); \quad i = 1, 2. \end{aligned} \quad (4.1) \quad \{\text{eq11}\}$$

Here  $\mathcal{R}$  denotes the set of all functions  $r(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  such that the last integral above converges. For  $i = 1, 2$  we define the *adjoint* processes  $p_i(t, y_1, y_2), q_i(t, y_1, y_2), r_i(t, y_1, y_2, \zeta)$  as the solution of the  $y_1, y_2$ -parametrised BSDEs

$$\begin{cases} dp_i(t, y_1, y_2) = -\frac{\partial H_i}{\partial x}(t, y_1, y_2) dt + q_i(t, y_1, y_2) dB(t) + \int_{\mathbb{R}} r_i(t, y_1, y_2, \zeta) \tilde{N}(dt, d\zeta); & 0 \leq t \leq T \\ p_i(T, y_1, y_2) = g'_i(x(T, y_1, y_2), y_1, y_2) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_T] \end{cases} \quad (4.2) \quad \{\text{eq12}\}$$

Let  $J_i(u(\cdot, y_1, y_2))$  be defined by

$$\begin{aligned} J_i(u(\cdot, y_1, y_2)) &= \mathbb{E}\left[\int_0^T f_i(t, x(t, y_1, y_2), u_1(t, y_1), u_2(t, y_2), y_1, y_2) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_t] dt \right. \\ &\quad \left. + g_i(x(T, y_1, y_2), y_1, y_2) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_T] \right] \end{aligned} \quad (4.3) \quad \{J(u)2\}$$

Then we see that

$$J_i(u_1, u_2) = \int_{\mathbb{R}} \int_{\mathbb{R}} J_i(u(\cdot, y_1, y_2)) dy_1 dy_2 \quad (4.4)$$

The insider game problem can therefore be written as follows:

**Problem 4.1** *For each  $y_1, y_2 \in \mathbb{R}$ , find  $(u_1^*(\cdot, y_1), u_2^*(\cdot, y_2)) \in \mathcal{A}_1 \times \mathcal{A}_2$  such that*

$$\sup_{u_1(\cdot, y_1) \in \mathcal{A}_1} \int_{\mathbb{R}} \int_{\mathbb{R}} J_1(u_1(\cdot, y_1), u_2^*(\cdot, y_2)) dy_1 dy_2 \leq \int_{\mathbb{R}} \int_{\mathbb{R}} J_1(u_1^*(\cdot, y_1), u_2^*(\cdot, y_2)) dy_1 dy_2 \quad (4.5) \quad \{\text{eq2.15}\}$$

and

$$\sup_{u_2(\cdot, y_2) \in \mathcal{A}_2} \int_{\mathbb{R}} \int_{\mathbb{R}} J_2(u_1^*(\cdot, y_1), u_2(\cdot, y_2)) dy_1 dy_2 \leq \int_{\mathbb{R}} \int_{\mathbb{R}} J_2(u_1^*(\cdot, y_1), u_2^*(\cdot, y_2)) dy_1 dy_2 \quad (4.6) \quad \{\text{eq2.16}\}$$

To study this problem we present two maximum principles for the corresponding games. The first is the following:

**Theorem 4.2** *[Sufficient maximum principle]*

*Let  $(\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$  with associated solution  $\hat{x}(t, y_1, y_2), \hat{p}_i(t, y_1, y_2), \hat{q}_i(t, y_1, y_2), \hat{r}_i(t, y_1, y_2, \zeta)$  of (3.6) and (4.2);  $i=1,2$ . Assume that the following hold:*

1.  $x \rightarrow g_i(x)$  is concave;  $i = 1, 2$

2. The functions

$$\hat{\mathcal{H}}_1(x) = \sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}} H_1(t, x, y_1, y_2, u_1, \hat{u}_2(t, y_2), \hat{p}_1(t, y_1, y_2), \hat{q}_1(t, y_1, y_2), \hat{r}_1(t, y_1, y_2, \cdot)) dy_2 \quad (4.7)$$

and

$$\hat{\mathcal{H}}_2(x) = \sup_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}} H_2(t, x, y_1, y_2, \hat{u}_1(t, y_1), u_2, \hat{p}_2(t, y_1, y_2), \hat{q}_2(t, y_1, y_2), \hat{r}_2(t, y_1, y_2, \cdot)) dy_1 \quad (4.8)$$

are concave for all  $t, y_1, y_2$

3.

$$\begin{aligned} & \sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}} H_1(t, \hat{x}(t, y_1, y_2), u_1, \hat{u}_2(t, y_2), \hat{p}_1(t, y_1, y_2), \hat{q}_1(t, y_1, y_2), \hat{r}_1(t, y_1, y_2, \cdot)) dy_2 \\ &= \int_{\mathbb{R}} H_1(t, \hat{x}(t, y_1, y_2), \hat{u}_1(t, y_1), \hat{u}_2(t, y_2), \hat{p}_1(t, y_1, y_2), \hat{q}_1(t, y_1, y_2), \hat{r}_1(t, y_1, y_2, \cdot)) dy_2 \\ & \text{for all } t, y_1. \end{aligned} \quad (4.9)$$

4.

$$\begin{aligned}
& \sup_{u_2 \in \mathbf{A}_2} \int_{\mathbb{R}} H_2(t, \hat{x}(t, y_1, y_2), \hat{u}_1(t, y_1), u_2, \hat{p}_2(t, y_1, y_2), \hat{q}_2(t, y_1, y_2), \hat{r}_2(t, y_1, y_2, \cdot)) dy_1 \\
&= \int_{\mathbb{R}} H_2(t, \hat{x}(t, y_1, y_2), \hat{u}_1(t, y_1), \hat{u}_2(t, y_2), \hat{p}_2(t, y_1, y_2), \hat{q}_2(t, y_1, y_2), \hat{r}_2(t, y_1, y_2, \cdot)) dy_1 \\
&\text{for all } t, y_2.
\end{aligned} \tag{4.10}$$

Then  $(u_1^*(\cdot, y_1), u_2^*(\cdot, y_2)) := (\hat{u}_1(\cdot, y_1), \hat{u}_2(\cdot, y_2))$  is a Nash equilibrium for the problem (4.5)-(4.6).

Proof. By considering an increasing sequence of stopping times  $\tau_n$  converging to  $T$ , we may assume that all local integrals appearing in the computations below are martingales and have expectation 0. We omit the details in this argument. See [ØS2].

We first prove that

$$\sup_{u_1(\cdot, y_1) \in \mathcal{A}_1} \int_{\mathbb{R}} \int_{\mathbb{R}} J_1(u_1(\cdot, y_1), \hat{u}_2(\cdot, y_2)) dy_1 dy_2 \leq \int_{\mathbb{R}} \int_{\mathbb{R}} J_1(\hat{u}_1(\cdot, y_1), \hat{u}_2(\cdot, y_2)) dy_1 dy_2 \quad (4.11) \quad \{\text{eq2.17}\}$$

Choose arbitrary  $u_1(\cdot, y_1) \in \mathcal{A}_1$  and let us in the following, for simplicity of notation, put  $x(t, y_1, y_2) = x^{u_1, \hat{u}_2}(t, y_1, y_2)$ ,  $\hat{x}(t, y_1, y_2) = x^{\hat{u}_1, \hat{u}_2}(t, y_1, y_2)$ ,  $b(t, y_1, y_2) = b(t, x(t, y_1, y_2), u_1(t, y_1), u_2(t, y_2), \omega)$ ,  $\hat{b}(t, y_1, y_2) = b(t, \hat{x}(t, y_1, y_2), u_1(t, y_1), \hat{u}_2(t, y_2), \omega)$  and similarly with  $\sigma(t, y_1, y_2)$ ,  $\hat{\sigma}(t, y_1, y_2)$ ,  $\gamma(t, y_1, y_2, \zeta)$ ,  $\hat{\gamma}(t, y_1, y_2, \zeta)$  and  $\tilde{x}(t, y_1, y_2) = x(t, y_1, y_2) - \hat{x}(t, y_1, y_2)$ . Let us also put

$$H_1(t, y_1, y_2) = H_1(t, x(t, y_1, y_2), y_1, y_2, u_1(t, y_1), \hat{u}_2(t, y_2), \hat{p}_1(t, y_1, y_2), \hat{q}_1(t, y_1, y_2), \hat{r}_1(t, y_1, y_2, \cdot)) \tag{4.12}$$

and

$$\hat{H}_1(t, y_1, y_2) = H_1(t, \hat{x}(t, y_1, y_2), y_1, y_2, \hat{u}_1(t, y_1), \hat{u}_2(t, y_2), \hat{p}_1(t, y_1, y_2), \hat{q}_1(t, y_1, y_2), \hat{r}_1(t, y_1, y_2, \cdot)) \tag{4.13}$$

Consider

$$\int_{\mathbb{R}} \int_{\mathbb{R}} [J_1(u_1(\cdot, y_1), \hat{u}_2(\cdot, y_2)) - J_1(\hat{u}_1(\cdot, y_1), \hat{u}_2(\cdot, y_2))] dy_1 dy_2 = I_1 + I_2,$$

where

$$\begin{aligned}
I_1 = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \Big[ \int_0^T \{ f_1(t, x(t, y_1, y_2), u_1(t, y_1), \hat{u}_2(t, y_2)) - f_1(t, \hat{x}(t, y_1, y_2), \hat{u}_1(t, y_1), \hat{u}_2(t, y_2)) \} \\
\mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_t] dt \Big] dy_1 dy_2
\end{aligned} \tag{4.14} \quad \{\text{I}_1\}$$

and

$$I_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} [\{ g_1(x(T, y_1, y_2)) - g_1(\hat{x}(T, y_1, y_2)) \} \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_T]] dy_1 dy_2. \tag{4.15} \quad \{\text{I}_2\}$$



By the definition of  $H_1$  we have

$$\begin{aligned} I_1 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T \{H_1(t, y_1, y_2) - \widehat{H}_1(t, y_1, y_2) - \widehat{p}_1(t, y_1, y_2) \widetilde{b}_1(t, y_1, y_2) - \widehat{q}_1(t, y_1, y_2) \widetilde{\sigma}(t, u_1, u_2, y_1, y_2) \right. \\ &\quad \left. - \int_{\mathbb{R}} \widehat{r}_1(t, y_1, y_2, \zeta) \widetilde{\gamma}(t, y_1, y_2, \zeta) \nu(d\zeta) \} dt \right] dy_1 dy_2. \end{aligned} \quad (4.16)$$

Since  $g_1$  is concave we have

$$\begin{aligned} I_2 &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}[g'_1(\widehat{x}(T, y_1, y_2)) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_T] \widetilde{x}(T, y_1, y_2)] dy_1 dy_2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}[\widehat{p}_1(T, y_1, y_2) \widetilde{x}(T, y_1, y_2)] dy_1 dy_2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T \widehat{p}_1(t, y_1, y_2) d\widetilde{x}(t, y_1, y_2) + \int_0^T \widetilde{x}(t, y_1, y_2) d\widehat{p}_1(t, y_1, y_2) + \int_0^T d[\widehat{p}_1, \widetilde{x}]_t \right] dy_1 dy_2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T \widehat{p}_1(t, y_1, y_2) (\widetilde{b}(t, y_1, y_2) dt + \widetilde{\sigma}(t, y_1, y_2) dB(t) + \int_{\mathbb{R}} \widetilde{\gamma}(t, y_1, y_2, \zeta) \widetilde{N}(dt, d\zeta)) \right. \\ &\quad \left. - \int_0^T \frac{\partial \widehat{H}_1}{\partial x}(t, y_1, y_2) \widetilde{x}(t, y_1, y_2) dt + \int_0^T \widehat{q}_1(t, y_1, y_2) \widetilde{x}(t, y_1, y_2) dB(t) \right. \\ &\quad \left. + \int_0^T \int_{\mathbb{R}} \widetilde{x}(t, y_1, y_2) \widehat{r}_1(t, y_1, y_2, \zeta) \widetilde{N}(dt, d\zeta) + \int_0^T \widetilde{\sigma}(t, y_1, y_2) \widehat{q}_1(t, y_1, y_2) dt \right. \\ &\quad \left. + \int_0^T \int_{\mathbb{R}} \widetilde{\gamma}(t, y_1, y_2, \zeta) \widehat{r}_1(t, y_1, y_2, \zeta) \nu(d\zeta) dt + \int_0^T \int_{\mathbb{R}} \widetilde{\gamma}(t, y_1, y_2, \zeta) \widehat{r}_1(t, y_1, y_2, \zeta) \widetilde{N}(dt, d\zeta) \right] dy_1 dy_2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T \widehat{p}_1(t, y_1, y_2) \widetilde{b}(t, y_1, y_2) dt - \int_0^T \frac{\partial \widehat{H}_1}{\partial x}(t, y_1, y_2) \widetilde{x}(t, y_1, y_2) dt + \int_0^T \widetilde{\sigma}(t, y_1, y_2) \widehat{q}_1(t, y_1, y_2) dt \right. \\ &\quad \left. + \int_0^T \int_{\mathbb{R}} \widetilde{\gamma}(t, y_1, y_2, \zeta) \widehat{r}_1(t, y_1, y_2, \zeta) \nu(d\zeta) dt \right] dy_1 dy_2. \end{aligned} \quad (4.17) \quad \{\text{II}_2\}$$

Adding (4.16) - (4.17) we get, by concavity of  $H_1$ ,

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} [J_1(u_1(\cdot, y_1), \widehat{u}_2(\cdot, y_2)) - J_1(\widehat{u}_1(\cdot, y_1), \widehat{u}_2(\cdot, y_2))] dy_1 dy_2 \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T \{H_1(t, y_1, y_2) - \widehat{H}_1(t, y_1, y_2) - \frac{\partial \widehat{H}_1}{\partial x}(t, y_1, y_2) \widetilde{x}(t, y_1, y_2)\} dt \right] dy_1 dy_2 \end{aligned} \quad (4.18) \quad \{\text{eq2.28}\}$$

Since  $\widehat{\mathcal{H}}_1(x)$  is concave, it follows by a standard separating hyperplane argument that there exists a supergradient  $a \in \mathbb{R}$  for  $\widehat{\mathcal{H}}_1(x)$  at  $x = \widehat{x}(t, y_1, y_2)$  such that if we define

$$\phi(x) = \widehat{\mathcal{H}}_1(x) - \widehat{\mathcal{H}}_1(\widehat{x}(t, y_1, y_2)) - a(x - \widehat{x}(t, y_1, y_2)) \quad (4.19)$$

then

$$\phi(x) \leq 0 \text{ for all } x \quad (4.20)$$

On the other hand, we clearly have

$$\phi(\hat{x}(t, y_1, y_2)) = 0 \quad (4.21)$$

it follows that

$$\frac{\partial \hat{\mathcal{H}}_1}{\partial x}(\hat{x}(t, y_1, y_2)) = \int_{\mathbb{R}} \frac{\partial \hat{H}_1}{\partial x}(t, \hat{x}(t, y_1, y_2), \hat{u}_1(t, y_1), \hat{u}_2(t, y_2), \hat{p}(t, y_1, y_2), \hat{q}(t, y_1, y_2), \hat{r}(t, y_1, y_2, \zeta)) dy_2 = a \quad (4.22)$$

Combining this with (4.18), we get

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} [J_1(u_1(\cdot, y_1), \hat{u}_2(\cdot, y_2)) - J_1(\hat{u}_1(\cdot, y_1), \hat{u}_2(\cdot, y_2))] dy_1 dy_2 \\ & \leq \int_0^T \int_{\mathbb{R}} [\hat{\mathcal{H}}_1(x(t, y_1, y_2)) - \hat{\mathcal{H}}_1(\hat{x}(t, y_1, y_2)) - \frac{\partial \hat{\mathcal{H}}_1}{\partial x}(\hat{x}(t, y_1, y_2))(x(t, y_1, y_2) - \hat{x}(t, y_1, y_2))] dy_1 dt \\ & \leq 0 \text{ since } \hat{\mathcal{H}}_1 \text{ is concave.} \end{aligned} \quad (4.23)$$

Hence

$$\sup_{u_1(\cdot, y_1) \in \mathcal{A}_1} \int_{\mathbb{R}} \int_{\mathbb{R}} J_1(u_1(\cdot, y_1), \hat{u}_2(\cdot, y_2)) dy_1 dy_2 \leq \int_{\mathbb{R}} \int_{\mathbb{R}} J_1(\hat{u}_1(\cdot, y_1), \hat{u}_2(\cdot, y_2)) dy_1 dy_2 \quad (4.24) \quad \{\text{eq2.17}\}$$

□

## 4.1 The case when only one of the players is an insider.

It is useful also to have a formulation in the partly degenerate case when only one of the players, say player number 1, has inside information. Then the control of player 1 is  $\mathbb{H}_1$ -adapted as before, while player 2 is  $\mathbb{F}$ -adapted. In this case we define the *Hamiltonians*  $H_i : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \Omega \rightarrow \mathbb{R}$  by

$$\begin{aligned} H_i(t, x, y_1, u_1, u_2, p, q, r) &= H(t, x, y_1, u_1, u_2, p, q, r, \omega) \\ &= \mathbb{E}[\delta_{Y_1}(y_1) | \mathcal{F}_t] f_i(t, x, u_1, u_2, y_1) + b(t, x, u_1, u_2, y_1) p \\ &\quad + \sigma(t, x, u_1, u_2, y_1) q + \int_{\mathbb{R}} \gamma(t, x, u_1, u_2, y_1) r(t, \zeta) \nu(d\zeta); i = 1, 2. \end{aligned} \quad (4.25) \quad \{\text{eq2.35}\}$$

Here, as before,  $\mathcal{R}$  denotes the set of all functions  $r(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  such that the last integral above converges. For  $i = 1, 2$  we define the *adjoint* processes  $p_i(t, y_1), q_i(t, y_1), r_i(t, y_1, \zeta)$  as the solution of the  $y_1$ -parametrised BSDEs

$$\begin{cases} dp_i(t, y_1) = -\frac{\partial H_i}{\partial x}(t, y_1) dt + q_i(t, y_1) dB(t) + \int_{\mathbb{R}} r_i(t, y_1, \zeta) \tilde{N}(dt, d\zeta); & 0 \leq t \leq T \\ p_i(T, y_1) = g'_i(x(T, y_1), y_1) \mathbb{E}[\delta_{Y_1}(y_1) | \mathcal{F}_T] \end{cases} \quad (4.26) \quad \{\text{eq2.36}\}$$

Let  $J_i(u(\cdot, y_1))$  be defined by

$$J_i(u(\cdot, y_1)) = \mathbb{E}\left[\int_0^T f_i(t, x(t, y_1), u_1(t, y_1), u_2(t), y_1) \mathbb{E}[\delta_{Y_1}(y_1) | \mathcal{F}_t] dt + g_i(x(T, y_1), y_1) \mathbb{E}[\delta_{Y_1}(y_1) | \mathcal{F}_T]\right] \quad (4.27) \quad \{\text{eq2.37}\}$$

Then we see that

$$J_i(u_1, u_2) = \int_{\mathbb{R}} \int_{\mathbb{R}} J_i(u(\cdot, y_1)) dy_1. \quad (4.28) \quad \{\text{eq2.38}\}$$

**Theorem 4.3** [Sufficient maximum principle with only one insider]

Suppose  $Y_2 = 0$ , i.e. player number 2 has no inside information. Let  $(\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$  with associated solution  $\hat{x}(t, y_1), \hat{p}_i(t, y_1), \hat{q}_i(t, y_1), \hat{r}_i(t, y_1, \zeta)$  of (3.6) and (4.2);  $i=1,2$ . Assume that the following hold:

1.  $x \rightarrow g_i(x)$  is concave;  $i = 1, 2$
2. The functions

$$\hat{\mathcal{H}}_1(x) = \sup_{u_1 \in \mathcal{A}_1} H_1(t, x, y_1, u_1, \hat{u}_2(t), \hat{p}_1(t, y_1), \hat{q}_1(t, y_1), \hat{r}_1(t, y_1, \cdot)) \quad (4.29)$$

and

$$\hat{\mathcal{H}}_2(x) = \sup_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}} H_2(t, x, y_1, \hat{u}_1(t, y_1), u_2, \hat{p}_2(t, y_1), \hat{q}_2(t, y_1), \hat{r}_2(t, y_1, \cdot)) dy_1 \quad (4.30)$$

are concave for all  $t$ .

- 3.

$$\begin{aligned} & \sup_{u_1 \in \mathcal{A}_1} H_1(t, \hat{x}(t, y_1), u_1, \hat{u}_2(t), \hat{p}_1(t, y_1), \hat{q}_1(t, y_1), \hat{r}_1(t, y_1, \cdot)) \\ &= H_1(t, \hat{x}(t, y_1), \hat{u}_1(t, y_1), \hat{u}_2(t), \hat{p}_1(t, y_1), \hat{q}_1(t, y_1), \hat{r}_1(t, y_1, \cdot)) \\ & \text{for all } t. \end{aligned} \quad (4.31)$$

- 4.

$$\begin{aligned} & \sup_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}} H_2(t, \hat{x}(t, y_1), \hat{u}_1(t, y_1), u_2, \hat{p}_2(t, y_1), \hat{q}_2(t, y_1), \hat{r}_2(t, y_1, \cdot)) dy_1 \\ &= \int_{\mathbb{R}} H_2(t, \hat{x}(t, y_1), \hat{u}_1(t, y_1), \hat{u}_2(t), \hat{p}_2(t, y_1), \hat{q}_2(t, y_1), \hat{r}_2(t, y_1, \cdot)) dy_1 \\ & \text{for all } t. \end{aligned} \quad (4.32)$$

Then  $(u_1^*(\cdot, y_1), u_2^*(\cdot)) := (\hat{u}_1(\cdot, y_1), \hat{u}_2(\cdot))$  is a Nash equilibrium for the problem (4.5)-(4.6).

Proof. The proof is similar to the proof of Theorem 2.2 and is omitted.  $\square$

## 5 A necessary maximum principle

We proceed to establish a corresponding necessary maximum principle. For this, we do not need concavity conditions, but in stead we need the following assumptions about the set of admissible control values:

- $A_1$ . For all  $t_0 \in [0, T]$ ,  $y_i \in \mathbb{R}$  and all bounded  $\mathcal{F}_{t_0}$ -measurable random variables  $\alpha_i(y_i, \omega)$ , the control  $\theta_i(t, y_i, \omega) := \mathbf{1}_{[t_0, T]}(t)\alpha_i(y_i, \omega)$  belongs to  $\mathcal{A}_i$  for  $i = 1, 2$ .

- $A_2$ . For all  $u_i; \beta_0^i \in \mathcal{A}_i$  with  $\beta_0^i(t, y_i) \leq K < \infty$  for all  $t, y_i$  define

$$\delta_i(t, y_i) = \frac{1}{2K} \text{dist}((u_i(t, y_i), \partial \mathbb{A}_i) \wedge 1 > 0 \quad (5.1) \quad \{\text{delta}\}$$

and put

$$\beta_i(t, y_i) = \delta_i(t, y_i)\beta_0^i(t, y_i). \quad (5.2) \quad \{\text{beta}(t, y)\}$$

Then there exists  $\delta > 0$  such that the control

$$\tilde{u}_i(t, y_i) = u_i(t, y_i) + a\beta_i(t, y_i); \quad t \in [0, T]$$

belongs to  $\mathcal{A}_i$  for all  $a \in (-\delta, \delta)$  for  $i = 1, 2$ .

- $A_3$ . For all  $\beta_i$  as in (5.2) the derivative processes

$$\chi_1(t, y_1, y_2) := \frac{d}{da} x^{(u_1 + a\beta_1, u_2)}(t, y_1, y_2)|_{a=0}$$

and

$$\chi_2(t, y_1, y_2) := \frac{d}{da} x^{(u_1, u_2 + a\beta_2)}(t, y_1, y_2)|_{a=0}$$

exists, and belong to  $\mathbf{L}^2(\lambda \times \mathbf{P})$  and

$$\begin{cases} d\chi_1(t, y_1, y_2) = [\frac{\partial b}{\partial x}(t, y_1, y_2)\chi_1(t, y_1, y_2) + \frac{\partial b}{\partial u_1}(t, y)\beta_1(t, y_1)]dt \\ + [\frac{\partial \sigma}{\partial x}(t, y_1, y_2)\chi_1(t, y) + \frac{\partial \sigma}{\partial u_1}(t, y_1, y_2)\beta_1(t, y_1)]dB(t) \\ + \int_{\mathbb{R}} [\frac{\partial \gamma}{\partial x}(t, y_1, y_2, \zeta)\chi_1(t, y_1, y_2) + \frac{\partial \gamma}{\partial u_1}(t, y_1, y_2, \zeta)\beta_1(t, y_1)]\tilde{N}(dt, d\zeta) \\ \chi_1(0, y_1, y_2) = \frac{d}{da} x^{(u_1 + a\beta_1, u_2)}(0, y_1, y_2)|_{a=0} = 0. \end{cases} \quad (5.3) \quad \{\text{d chi}\}$$

and

$$\begin{cases} d\chi_2(t, y) = [\frac{\partial b}{\partial x}(t, y_1, y_2)\chi_2(t, y_1, y_2) + \frac{\partial b}{\partial u_2}(t, y_1, y_2)\beta_2(t, y_2)]dt \\ + [\frac{\partial \sigma}{\partial x}(t, y_1, y_2)\chi_2(t, y_1, y_2) + \frac{\partial \sigma}{\partial u_2}(t, y_1, y_2)\beta_2(t, y_2)]dB(t) \\ + \int_{\mathbb{R}} [\frac{\partial \gamma}{\partial x}(t, y_1, y_2, \zeta)\chi_2(t, y_1, y_2) + \frac{\partial \gamma}{\partial u_2}(t, y_1, y_2, \zeta)\beta_2(t, y_2)]\tilde{N}(dt, d\zeta) \\ \chi(0, y_1, y_2) = \frac{d}{da} x^{(u_1, u_2 + a\beta_2)}(0, y_1, y_2)|_{a=0} = 0. \end{cases} \quad (5.4) \quad \{\text{d chi2}\}$$

**Theorem 5.1** *[Necessary maximum principle]*

Let  $(u_1, u_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ . Then the following are equivalent:

1.  $\frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J_1(u_1 + a\beta_1, u_2)|_{a=0} dy_1 dy_2 = \frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J_2(u_1, u_2 + a\beta_2)|_{a=0} dy_1 dy_2 = 0$  for all bounded  $\beta_i \in \mathcal{A}_i$  of the form (5.2).

2.

$$\begin{aligned} & \left[ \int_{\mathbb{R}} \frac{\partial H_1}{\partial v_1}(t, x(t, y_1, y_2), v_1, u_2(t, y_2), p_1(t, y_1, y_2), q_1(t, y_1, y_2), r_1(t, y_1, y_2, \cdot)) dy_2 \right]_{v_1=u_1(t, y_1)} \\ &= \left[ \int_{\mathbb{R}} \frac{\partial H_2}{\partial v_2}(t, x(t, y_1, y_2), u_1(t, y_1), v_2, p_2(t, y_1, y_2), q_2(t, y_1, y_2), r_2(t, y_1, y_2, \cdot)) dy_1 \right]_{v_2=u_2(t, y_2)} \\ &= 0 \quad \forall t \in [0, T] \end{aligned} \tag{5.5}$$

Proof. By considering an increasing sequence of stopping times  $\tau_n$  converging to  $T$ , we may assume that all local integrals appearing in the computations below are martingales and have expectation 0. See [ØS2].

We can write

$$\frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J_1(u_1 + a\beta_1, u_2)|_{a=0} dy_1 dy_2 = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= \frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T f_1(t, x^{u_1+a\beta_1}(t, y_1, y_2), u_1(t, y_1) + a\beta_1(t, y_1), u_2(t, y_2), y_1, y_2) \right. \\ &\quad \left. \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_t] dt \right] |_{a=0} dy_1 dy_2 \end{aligned} \tag{5.6}$$

and

$$I_2 = \frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}[g_1(x^{(u_1+a\beta_1, u_2)}(T, y_1, y_2), y_1, y_2) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_T]] |_{a=0} dy_1 dy_2.$$

By our assumptions on  $f_1$  and  $g_1$  and by (4.2) we have

$$I_1 = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T \left\{ \frac{\partial f_1}{\partial x}(t, y_1, y_2) \chi_1(t, y_1, y_2) + \frac{\partial f_1}{\partial u_1}(t, y_1, y_2) \beta_1(t, y_1) \right\} \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_t] dt \right] dy_1 dy_2 \tag{5.7} \quad \{\text{iii1}\}$$

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}[g'_1(x(T, y_1, y_2), y_1, y_2) \chi_1(T, y_1, y_2) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_T]] dy_1 dy_2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}[p_1(T, y_1, y_2) \chi_1(T, y_1, y_2)] dy_1 dy_2 \end{aligned} \tag{5.8} \quad \{\text{iii2}\}$$

By the Itô formula

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}[p_1(T, y_1, y_2) \chi_1(T, y_1, y_2)] dy_1 dy_2 \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T p_1(t, y_1, y_2) d\chi_1(t, y_1, y_2) + \int_0^T \chi_1(t, y_1, y_2) dp_1(t, y_1, y_2) + \int_0^T d[\chi_1, p_1](t, y_1, y_2) \right] dy_1 dy_2 \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T p_1(t, y_1, y_2) \left\{ \frac{\partial b}{\partial x}(t, y_1, y_2) \chi_1(t, y_1, y_2) + \frac{\partial b}{\partial u_1}(t, y_1, y_2) \beta_1(t, y_1) \right\} dt \right. \tag{5.9}
\end{aligned}$$

$$\begin{aligned}
&+ \int_0^T p_1(t, y_1, y_2) \left\{ \frac{\partial \sigma}{\partial x}(t, y_1, y_2) \chi_1(t, y_1, y_2) + \frac{\partial \sigma}{\partial u_1}(t, y_1, y_2) \beta_1(t, y_1) \right\} dB(t) \\
&+ \int_0^T \int_{\mathbb{R}} p_1(t, y_1, y_2) \left\{ \frac{\partial \gamma}{\partial x}(t, y_1, y_2, \zeta) \chi_1(t, y_1, y_2) + \frac{\partial \gamma}{\partial u_1}(t, y_1, y_2, \zeta) \beta_1(t, y_1) \right\} \tilde{N}(dt, d\zeta) \\
&- \int_0^T \chi_1(t, y_1, y_2) \frac{\partial H_1}{\partial x}(t, y_1, y_2) dt + \int_0^T \chi_1(t, y_1, y_2) q_1(t, y_1, y_2) dB(t) \\
&+ \int_0^T \int_{\mathbb{R}} \chi_1(t, y_1, y_2) r_1(t, y_1, y_2, \zeta) \tilde{N}(dt, d\zeta) \tag{5.10} \\
&+ \int_0^T q_1(t, y_1, y_2) \left\{ \frac{\partial \sigma}{\partial x}(t, y_1, y_2) \chi_1(t, y_1, y_2) + \frac{\partial \sigma}{\partial u_1}(t, y_1, y_2) \beta_1(t, y_1) \right\} dt \\
&+ \int_0^T \int_{\mathbb{R}} \left\{ \frac{\partial \gamma}{\partial x}(t, y_1, y_2, \zeta) \chi_1(t, y_1, y_2) + \frac{\partial \gamma}{\partial u_1}(t, y_1, y_2, \zeta) \beta_1(t, y_1) \right\} r_1(t, y_1, y_2, \zeta) \nu(d\zeta) dt \Big] dy_1 dy_2 \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T \chi_1(t, y_1, y_2) \left\{ p_1(t, y_1, y_2) \frac{\partial b}{\partial x}(t, y_1, y_2) + q_1(t, y_1, y_2) \frac{\partial \sigma}{\partial x}(t, y_1, y_2) - \frac{\partial H_1}{\partial x}(t, y_1, y_2) \right. \right. \\
&+ \int_{\mathbb{R}} \frac{\partial \gamma}{\partial x}(t, y_1, y_2, \zeta) r_1(t, y_1, y_2, \zeta) \nu(d\zeta) \Big\} dt \\
&+ \int_0^T \beta_1(t, y_1) \left\{ p_1(t, y_1, y_2) \frac{\partial b}{\partial u_1}(t, y_1, y_2) + q_1(t, y_1, y_2) \frac{\partial \sigma}{\partial u_1}(t, y_1, y_2) \right. \\
&+ \left. \left. \int_{\mathbb{R}} \frac{\partial \gamma}{\partial u_1}(t, y_1, y_2, \zeta) r_1(t, y_1, y_2, \zeta) \nu(d\zeta) \right\} dt \right] dy_1 dy_2 \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ - \int_0^T \chi_1(t, y_1, y_2) \frac{\partial f_1}{\partial x} \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_t] dt \right. \\
&+ \left. \int_0^T \left\{ \frac{\partial H_1}{\partial u_1}(t, y_1, y_2) - \frac{\partial f_1}{\partial u_1}(t, y_1, y_2) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_t] \right\} \beta_1(t, y_1) dt \right] dy_1 dy_2 \\
&= -I_1 + \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T \frac{\partial H_1}{\partial u_1}(t, y_1, y_2) \beta_1(t, y_1) dt \right] dy_1 dy_2. \tag{5.11}
\end{aligned}$$

Summing (5.7) and (5.9) we get

$$\frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J_1(u_1 + a\beta_1, u_2)|_{a=0} dy_1 dy_2 = I_1 + I_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T \frac{\partial H_1}{\partial u_1}(t, y_1, y_2) \beta_1(t, y_1) dt \right] dy_1 dy_2.$$

we conclude that

$$\frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J_1(u_1 + a\beta_1, u_2)|_{a=0} dy_1 dy_2 = 0$$

if and only if

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T \frac{\partial H_1}{\partial u_1}(t, y_1, y_2) \beta_1(t, y_1) dt \right] dy_1 dy_2 = 0$$

for all bounded  $\beta_1 \in \mathcal{A}_1$  of the form (5.2).

Changing the order of integration we can write this as follows:

$$\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}} F_1(t, y_1) \beta_1(t, y_1) dt dy_1 \right] = 0, \quad \forall \beta_1 \in \mathcal{A}_1 \quad (5.12)$$

where

$$F_1(t, y_1) := \int_{\mathbb{R}} \frac{\partial H_1}{\partial u_1}(t, y_1, y_2) dy_2. \quad (5.13)$$

In particular, applying this to  $\beta_1(t, y_1) = \theta_1(t, y_1)$  as in A1, we get that this is again equivalent to

$$\mathbb{E}[F_1(t, y_1) | \mathcal{F}_t] = 0, \quad \forall t, y_1.$$

Since  $F_1(t, y_1)$  is already  $\mathcal{F}_t$ -adapted, we have

$$\mathbb{E}[F_1(t, y_1) | \mathcal{F}_t] = F_1(t, y_1), \quad \forall t, y_1.$$

So we deduce that

$$F_1(t, y_1) = \int_{\mathbb{R}} \frac{\partial H_1}{\partial u_1}(t, y_1, y_2) dy_2 = 0, \quad \forall t, y_1.$$

A similar argument gives that

$$\frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J_2(u_1, u_2 + a\beta_2)|_{a=0} dy_1 dy_2 = 0 \text{ for all bounded } \beta_2 \in \mathcal{A}_2 \quad (5.14)$$

is equivalent to

$$\int_{\mathbb{R}} \frac{\partial H_2}{\partial u_2}(t, y_1, y_2) dy_1 = 0, \quad \forall t, y_2, \quad (5.15)$$

where

$$\begin{aligned} & \frac{\partial H_2}{\partial u_2}(t, y_1, y_2) \\ &= \frac{\partial H_2}{\partial v_2}(t, x(t, y_1, y_2), u_1(t, y_1), v_2, p_2(t, y_1, y_2), q_2(t, y_1, y_2), r_2(t, y_1, y_2, \cdot))_{v_2=u_2(t, y_2)}). \end{aligned} \quad (5.16)$$

□

## 6 The zero-sum game case

In the zero-sum case we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} J_1(u_1(., y_1), u_2(., y_2)) + J_2(u_1(., y_1), u_2(., y_2)) dy_1 dy_2 = 0. \quad (6.1)$$

Then the Nash equilibrium  $(\hat{u}_1(., y_1), \hat{u}_2(., y_2)) \in \mathcal{A}_1 \times \mathcal{A}_2$  satisfying (4.5)-(4.6) becomes a saddle point for

$$\int_{\mathbb{R}} \int_{\mathbb{R}} J(u_1(., y_1), u_2(., y_2)) dy_1 dy_2 := \int_{\mathbb{R}} \int_{\mathbb{R}} J_1(u_1(., y_1), u_2(., y_2)) dy_1 dy_2. \quad (6.2)$$

To see this, note that (4.5)-(4.6) imply that

$$\begin{aligned} \int_{\mathbb{R}^2} J_1(u_1(., y_1), \hat{u}_2(., y_2)) dy_1 dy_2 &\leq \int_{\mathbb{R}^2} J_1(\hat{u}_1(., y_1), \hat{u}_2(., y_2)) dy_1 dy_2 = - \int_{\mathbb{R}^2} J_2(\hat{u}_1(., y_1), \hat{u}_2(., y_2)) dy_1 dy_2 \\ &\leq - \int_{\mathbb{R}^2} J_2(\hat{u}_1(., y_1), u_2(., y_2)) dy_1 dy_2 \end{aligned} \quad (6.3)$$

and hence

$$\int_{\mathbb{R}^2} J(u_1(., y_1), \hat{u}_2(., y_2)) dy_1 dy_2 \leq \int_{\mathbb{R}^2} J(\hat{u}_1(., y_1), \hat{u}_2(., y_2)) dy_1 dy_2 \leq \int_{\mathbb{R}^2} J(\hat{u}_1(., y_1), u_2(., y_2)) dy_1 dy_2 \quad (6.4)$$

for all  $u_1, u_2$ . From this we deduce that

$$\begin{aligned} \inf_{u_2 \in \mathcal{A}_2} \sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}^2} J(u_1(., y_1), u_2(., y_2)) dy_1 dy_2 &\leq \sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}^2} J(u_1(., y_1), \hat{u}_2(., y_2)) dy_1 dy_2 \\ &\leq \int_{\mathbb{R}^2} J(\hat{u}_1(., y_1), \hat{u}_2(., y_2)) dy_1 dy_2 \leq \inf_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}^2} J(\hat{u}_1(., y_1), u_2(., y_2)) dy_1 dy_2 \\ &\leq \sup_{u_1 \in \mathcal{A}_1} \inf_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}^2} J(u_1(., y_1), u_2(., y_2)) dy_1 dy_2. \end{aligned} \quad (6.5)$$

Since we always have  $\inf \sup \geq \sup \inf$ , we conclude that

$$\begin{aligned} \inf_{u_2 \in \mathcal{A}_2} \sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}^2} J(u_1(., y_1), u_2(., y_2)) dy_1 dy_2 &= \sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}^2} J(u_1(., y_1), \hat{u}_2(., y_2)) dy_1 dy_2 \\ &= \int_{\mathbb{R}^2} J(\hat{u}_1(., y_1), \hat{u}_2(., y_2)) dy_1 dy_2 = \inf_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}^2} J(\hat{u}_1(., y_1), u_2(., y_2)) dy_1 dy_2 \\ &= \sup_{u_1 \in \mathcal{A}_1} \inf_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}^2} J(u_1(., y_1), u_2(., y_2)) dy_1 dy_2 \end{aligned} \quad (6.6)$$



i.e  $(\hat{u}_1(., y_1), \hat{u}_2(., y_2)) \in \mathcal{A}_1 \times \mathcal{A}_2$  is a saddle point for  $\int_{\mathbb{R}} \int_{\mathbb{R}} J(u_1(., y_1), u_2(., y_2)) dy_1 dy_2$ . Hence we want to find  $(\hat{u}_1(., y_1), \hat{u}_2(., y_2)) \in \mathcal{A}_1 \times \mathcal{A}_2$  such that

$$\begin{aligned} \sup_{u_1 \in \mathcal{A}_1} \inf_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}^2} J(u_1(., y_1), u_2(., y_2)) dy_1 dy_2 &= \inf_{u_2 \in \mathcal{A}_2} \sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}^2} J(u_1(., y_1), u_2(., y_2)) dy_1 dy_2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} J(\hat{u}_1(., y_1), \hat{u}_2(., y_2)) dy_1 dy_2 \end{aligned} \quad (6.7)$$

where

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} J(u(., y_1, y_2)) dy_1 dy_2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^T f(t, x(t, y_1, y_2), u_1(t, y_1), u_2(t, y_2), y_1, y_2) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_t] dt \right. \\ &\quad \left. + g(x(T, y_1, y_2), y_1, y_2) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_T] \right] dy_1 dy_2 \end{aligned} \quad (6.8) \quad \{J(u)\}$$

## 6.1 Situation 1: Both players are still maximising their own performance functional

Choose  $g_1 = g = -g_2$  and  $f_1 = f = -f_2$ . Then by (6.12) the Hamiltonians are:

$$\begin{aligned} H_1(t, x, y_1, y_2, u_1, u_2, p, q, r) &= H_1(t, x, y_1, y_2, u_1, u_2, p, q, r, \omega) \\ &= \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_t] f(t, x, u_1, u_2, y_1, y_2) + b(t, x, u_1, u_2, y_1, y_2) p \\ &\quad + \sigma(t, x, u_1, u_2, y_1, y_2) q + \int_{\mathbb{R}} \gamma(t, x, u_1, u_2, y_1, y_2) r(t, \zeta) \nu(d\zeta) \end{aligned} \quad (6.9) \quad \{\text{eq4.11}\}$$

and

$$\begin{aligned} H_2(t, x, y_1, y_2, u_1, u_2, p, q, r) &= H_2(t, x, y_1, y_2, u_1, u_2, p, q, r, \omega) \\ &= -\mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_t] f(t, x, u_1, u_2, y_1, y_2) + b(t, x, u_1, u_2, y_1, y_2) p \\ &\quad + \sigma(t, x, u_1, u_2, y_1, y_2) q + \int_{\mathbb{R}} \gamma(t, x, u_1, u_2, y_1, y_2) r(t, \zeta) \nu(d\zeta) \end{aligned} \quad (6.10) \quad \{\text{eq4.12}\}$$

Let  $p_i, q_i, r_i, i = 1, 2$  be as in (4.2).

Let us now state the necessary maximum principle for the zero sum game problem:

**Theorem 6.1** [Necessary maximum principle for zero-sum games]

Assume the conditions of Theorem 5.1 hold. Then the following are equivalent:

1.  $\frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J(u_1 + a\beta_1, u_2) |_{a=0} dy_1 dy_2 = \frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J(u_1, u_2 + a\beta_2) |_{a=0} dy_1 dy_2 = 0$  for all bounded  $\beta_i \in \mathcal{A}_i$  of the form (5.2).

2.

$$\begin{aligned} & \left[ \int_{\mathbb{R}} \frac{\partial H_1}{\partial v_1}(t, x(t, y_1, y_2), v_1, u_2(t, y_2), p_1(t, y_1, y_2), q_1(t, y_1, y_2), r_1(t, y_1, y_2, \cdot)) dy_2 \right]_{v_1=u_1(t, y_1)} \\ &= \left[ \int_{\mathbb{R}} \frac{\partial H_2}{\partial v_2}(t, x(t, y_1, y_2), u_1(t, y_1), v_2, p_2(t, y_1, y_2), q_2(t, y_1, y_2), r_2(t, y_1, y_2, \cdot)) dy_1 \right]_{v_2=u_2(t, y_2)} \\ &= 0 \quad \forall t \in [0, T], y_1, y_2. \end{aligned} \quad (6.11)$$

Proof. This is a direct consequence of Theorem 5.1 □

## 6.2 Situation 2: One of the players is maximising and the other is minimising the performance functional

Let us now look at the problem with one performance functional common to both players, but where one of the players is maximising and the other is minimising it. Then we get just one Hamiltonian and just one BSDE, which is simpler to deal with.

In this case the Hamiltonian  $H$ , is given by:

$$\begin{aligned} H(t, x, y_1, y_2, u_1, u_2, p, q, r) &= H(t, x, y_1, y_2, u_1, u_2, p, q, r, \omega) \\ &= \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_t] f(t, x, u_1, u_2, y_1, y_2) + b(t, x, u_1, u_2, y_1, y_2) p \\ &\quad + \sigma(t, x, u_1, u_2, y_1, y_2) q + \int_{\mathbb{R}} \gamma(t, x, u_1, u_2, y_1, y_2) r(t, \zeta) \nu(d\zeta) \end{aligned} \quad (6.12) \quad \{\text{eq11}\}$$

Moreover, there is only one triple  $(p, q, r)$  of adjoint processes, given by the BSDE

$$\begin{cases} dp(t, y_1, y_2) = -\frac{\partial H}{\partial x}(t, y_1, y_2) dt + q(t, y_1, y_2) dB(t) + \int_{\mathbb{R}} r(t, y_1, y_2, \zeta) \tilde{N}(dt, d\zeta); & 0 \leq t \leq T \\ p(T, y) = g'(x(T, y_1, y_2), y_1, y_2) \mathbb{E}[\delta_{Y_1, Y_2}(y_1, y_2) | \mathcal{F}_T] \end{cases} \quad (6.13) \quad \{\text{eq4.12}\}$$

We can now state the corresponding sufficient maximum principle for the zero-sum game:

**Theorem 6.2** (*Sufficient maximum principle for the zero-sum game*)

Let  $(\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$  with associated solution  $\hat{x}(t, y_1, y_2), \hat{p}(t, y_1, y_2), \hat{q}(t, y_1, y_2), \hat{r}(t, y_1, y_2, \zeta)$  of (3.6) and (6.13). Assume that the following holds:

1. the function  $x \rightarrow g(x)$  is affine
- 2.

$$\begin{aligned} &\sup_{u_1 \in \mathbf{A}_1} \int_{\mathbb{R}} H(t, \hat{x}(t, y_1, y_2), u_1, \hat{u}_2(t, y_2), \hat{p}(t, y_1, y_2), \hat{q}(t, y_1, y_2), \hat{r}(t, y_1, y_2, \cdot)) dy_2 \\ &= \int_{\mathbb{R}} H(t, \hat{x}(t, y_1, y_2), \hat{u}_1(t, y_1), \hat{u}_2(t, y_2), \hat{p}(t, y_1, y_2), \hat{q}(t, y_1, y_2), \hat{r}(t, y_1, y_2, \cdot)) dy_2 \\ &\text{for all } t, y_1. \end{aligned} \quad (6.14)$$

$$\begin{aligned} &\inf_{u_2 \in \mathbf{A}_2} \int_{\mathbb{R}} H(t, \hat{x}(t, y_1, y_2), \hat{u}_1(t, y_1), u_2, \hat{p}(t, y_1, y_2), \hat{q}(t, y_1, y_2), \hat{r}(t, y_1, y_2, \cdot)) dy_1 \\ &= \int_{\mathbb{R}} H(t, \hat{x}(t, y_1, y_2), \hat{u}_1(t, y_1), \hat{u}_2(t, y_2), \hat{p}(t, y_1, y_2), \hat{q}(t, y_1, y_2), \hat{r}(t, y_1, y_2, \cdot)) dy_1 \\ &\text{for all } t, y_2. \end{aligned} \quad (6.15)$$

### 3. The function

$$\hat{\mathcal{H}}(x) = \sup_{u_1 \in \mathcal{A}_1} \int_{\mathbb{R}} H(t, x, y_1, y_2, u_1, \hat{u}_2(t, y_2), \hat{p}(t, y_1, y_2), \hat{q}(t, y_1, y_2), \hat{r}(t, y_1, y_2, \cdot)) dy_2 \quad (6.16)$$

is concave for all  $t, y_1$ ,  
and the function

$$\underline{\mathcal{H}}(x) = \inf_{u_2 \in \mathcal{A}_2} \int_{\mathbb{R}} H(t, x, y_1, y_2, \hat{u}_1(t, y_1), u_2, \hat{p}(t, y_1, y_2), \hat{q}(t, y_1, y_2), \hat{r}(t, y_1, y_2, \cdot)) dy_1 \quad (6.17)$$

is convex for all  $t, y_2$ .

Then  $\hat{u}(t, y_1, y_2) = (\hat{u}_1(t, y_1), \hat{u}_2(t, y_2))$  is a saddle point for  $J(u_1, u_2)$ .

Let us now state the necessary maximum principle for the zero sum game problem:

**Theorem 6.3** [Necessary maximum principle for zero-sum games]

Assume the conditions of Theorem 5.1 hold. Then the following are equivalent:

$$1. \frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J(u_1 + a\beta_1, u_2)|_{a=0} dy_1 dy_2 = \frac{d}{da} \int_{\mathbb{R}} \int_{\mathbb{R}} J(u_1, u_2 + a\beta_2)|_{a=0} dy_1 dy_2 = 0 \text{ for all bounded } \beta_i \in \mathcal{A}_i \text{ of the form (5.2).}$$

2.

$$\begin{aligned} & \left[ \int_{\mathbb{R}} \frac{\partial H}{\partial v_1}(t, x(t, y_1, y_2), v_1, u_2(t, y_2), p_1(t, y_1, y_2), q_1(t, y_1, y_2), r_1(t, y_1, y_2, \cdot)) dy_2 \right]_{v_1 = u_1(t, y_1)} \\ &= \left[ \int_{\mathbb{R}} \frac{\partial H}{\partial v_2}(t, x(t, y_1, y_2), u_1(t, y_1), v_2, p_2(t, y_1, y_2), q_2(t, y_1, y_2), r_2(t, y_1, y_2, \cdot)) dy_1 \right]_{v_2 = u_2(t, y_2)} \\ &= 0 \quad \forall t \in [0, T], y_1, y_2. \end{aligned} \quad (6.18)$$

## 7 Applications

### 7.1 Optimal insider consumption under model uncertainty

Suppose we have a cash flow with consumption, modelled by the process  $X(t, Y) = X^{c, \mu}(t, Y)$  defined by:

$$\begin{cases} dX(t, Y) = (\alpha(t, Y) + \mu(t, Y_2) - c(t, Y_1))X(t, Y)dt + \beta(t, Y)X(t, Y)dB(t) + \int_{\mathbb{R}} \gamma(t, Y, \zeta)X(t, Y)\tilde{N}(dt, d\zeta) \\ X(0) = x > 0 \end{cases}$$

Here  $\alpha(t, Y), \beta(t, Y), \gamma(t, Y)$  are given coefficients, while  $c(t, Y_1) > 0$  is the relative consumption rate chosen by the consumer (player number 1) and  $\mu(t, Y_2)$  is a perturbation of the

drift term, representing the model uncertainty chosen by the environment (player number 2). Define the performance functional by

$$J(c, \mu) = \mathbb{E}\left[\int_0^T \{\log(c(t)X(t)) + \frac{1}{2}\mu^2(t)\}dt + \theta \log X(T)\right] \quad (7.1)$$

where  $\theta > 0$  is a given constant and  $\frac{1}{2}\mu^2(t)$  represents a penalty rate, penalizing  $\mu$  for being away from 0. We assume that  $c$  is  $\mathbb{H}^1$ -adapted, while  $\mu$  is  $\mathbb{H}^2$ -adapted.

We want to find  $c^* \in \mathcal{A}_1$  and  $\mu^* \in \mathcal{A}_2$  such that

$$\sup_{c \in \mathcal{A}_1} \inf_{\mu \in \mathcal{A}_2} J(c, \mu) = J(c^*, \mu^*). \quad (7.2) \quad \{\text{eq0.2}\}$$

As before we rewrite this problem as a classical stochastic differential game with two parameters  $y_1, y_2$ . Thus we define, for  $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}$ ,

$$\begin{cases} dx(t, y) &= (\alpha(t, y) + \mu(t, y_2) - c(t, y_1))x(t, y)dt + \beta(t, y)x(t, y)dB(t) \\ &+ \int_{\mathbb{R}} \gamma(t, y, \zeta)x(t, y)\tilde{N}(dt, d\zeta) \\ x(0, y) &= x > 0 \end{cases} \quad (7.3)$$

and

$$\begin{aligned} &J(c(\cdot, y_1), \mu(\cdot, y_2)) \\ &= \mathbb{E}\left[\int_0^T \{\log(c(t, y_1)x(t, y)) + \frac{1}{2}\mu^2(t, y_2)\}\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dt + \theta \log x(T, y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]\right] \end{aligned} \quad (7.4)$$

The Hamiltonian for this problem is

$$\begin{aligned} &H(t, x, y, c, \mu, p, q, r) \\ &= \{\log(cx) + \frac{1}{2}\mu^2\}\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] + (\alpha(t, y) + \mu - c)xp + \beta(t, y)xq + x \int_{\mathbb{R}} \gamma(t, y, \zeta)r(\zeta)d\nu(\zeta) \end{aligned} \quad (7.5)$$

and the BSDE for the adjoint processes  $p, q, r$  is

$$\begin{cases} dp(t, y) &= -[\frac{1}{x(t, y)}\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] + (\alpha(t, y) + \mu(t, y_2) - c(t, y_1))p(t, y) \\ &+ \beta(t, y)q(t, y) + \int_{\mathbb{R}} \gamma(t, y, \zeta)r(\zeta)d\nu(\zeta)]dt \\ &+ q(t, y)dB(t) + \int_{\mathbb{R}} r(t, y)\tilde{N}(dt, d\zeta); 0 \leq t \leq T \\ p(T, y) &= \frac{\theta}{x(T, y)}\mathbb{E}[\delta_Y(y)|\mathcal{F}_T] \end{cases} \quad (7.6) \quad \{\text{eq0.6}\}$$

Define

$$h(t, y) = p(t, y)x(t, y). \quad (7.7) \quad \{\text{eq0.7}\}$$

Then by the Itô formula we get

$$\begin{aligned}
dh(t, y) = & x(t, y) \left[ -\frac{1}{x(t, y)} \mathbb{E}[\delta_Y(y) | \mathcal{F}_t] - (\alpha(t, Y) + \mu(t, Y_2) - c(t, Y_1))p(t, y) - \beta(t, y)q(t, y) \right. \\
& - \int_{\mathbb{R}} \gamma(t, y, \zeta)r(t, \zeta)d\nu(\zeta) \Big] dt \\
& + p(t, y)(\alpha(t, Y) + \mu(t, Y_2) - c(t, Y_1))x(t, y)dt + p(t, y)\beta(t, y)x(t, y)dB(t) + x(t, y)q(t, y)dB(t) \\
& + q(t, y)\beta(t, y)x(t, y)dt \\
& + \int_{\mathbb{R}} [(x(t, y) + \gamma(t, y, \zeta)x(t, y))(p(t, y) + r(t, y, \zeta)) - p(t, y)x(t, y) - p(t, y)\gamma(t, y, \zeta)x(t, y) - x(t, y)r(t, y, \zeta)] \tilde{N}(dt, d\zeta)
\end{aligned} \tag{7.8} \quad \{\text{eq0.8}\}$$

$$\begin{aligned}
& + \int_{\mathbb{R}} [(x(t, y) + \gamma(t, y, \zeta)x(t, y))(p(t, y) + r(t, y, \zeta)) - p(t, y)x(t, y)] \tilde{N}(dt, d\zeta) \\
& = dF(t, y) + h(t, y)\beta(t, y)dB(t) + h(t, y) \int_{\mathbb{R}} \gamma(t, y, \zeta)\tilde{N}(dt, d\zeta),
\end{aligned} \tag{7.9}$$

where

$$\begin{aligned}
dF(t, y) = & - \mathbb{E}[\delta_Y(y) | \mathcal{F}_t]dt + x(t, y)q(t, y)dB(t) + x(t, y) \int_{\mathbb{R}} r(t, y, \zeta)(1 + \gamma(t, y, \zeta))\tilde{N}(dt, d\zeta).
\end{aligned} \tag{7.10}$$

To simplify this, we define the process  $k(t, y)$  by the equation

$$dk(t, y) = k(t, y) \left[ b(t, y)dB(t) + \int_{\mathbb{R}} c(t, y, \zeta)\tilde{N}(dt, d\zeta) \right] \tag{7.11} \quad \{\text{eq0.11}\}$$

for suitable processes  $b, c$  (to be determined).

Then again by the Itô formula we get

$$\begin{aligned}
d(h(t, y)k(t, y)) = & h(t, y)k(t, y) \left[ b(t, y)dB(t) + \int_{\mathbb{R}} c(t, y, \zeta)\tilde{N}(dt, d\zeta) \right] \\
& + k(t, y) \left[ dF(t, y) + h(t, y)\beta(t, y)dB(t) + h(t, y) \int_{\mathbb{R}} \gamma(t, y, \zeta)\tilde{N}(dt, d\zeta) \right] \\
& + (h(t, y)\beta(t, y) + x(t, y)q(t, y))k(t, y)b(t, y)dt \\
& + \int_{\mathbb{R}} \left( h(t, y)\gamma(t, y, \zeta) + x(t, y)r(t, y, \zeta)(1 + \gamma(t, y, \zeta)) \right) k(t, y)c(t, y, \zeta)\tilde{N}(dt, d\zeta) \\
& + \int_{\mathbb{R}} \left( h(t, y)\gamma(t, y, \zeta) + x(t, y)r(t, y, \zeta)(1 + \gamma(t, y, \zeta)) \right) k(t, y)c(t, y, \zeta)d\nu(\zeta)dt
\end{aligned} \tag{7.12} \quad \{\text{eq0.12}\}$$

Define

$$u(t, y) := h(t, y)k(t, y). \tag{7.13} \quad \{\text{eq0.13}\}$$

Then the equation above can be written

$$\begin{aligned}
du(t, y) = & u(t, y) \left[ \int_{\mathbb{R}} \gamma(t, y, \zeta) c(t, y, \zeta) d\nu(\zeta) dt \right. \\
& + \{ \beta(t, y) + b(t, y) \} dB(t) + \beta(t, y) b(t, y) dt + \int_{\mathbb{R}} \{ c(t, y, \zeta) + \gamma(t, y, \zeta) + c(t, y, \zeta) \gamma(t, y, \zeta) \} \tilde{N}(dt, d\zeta) \\
& + k(t, y) \left[ dF(t, y) + x(t, y) q(t, y) b(t, y) dt + \int_{\mathbb{R}} x(t, y) r(t, y, \zeta) c(t, y, \zeta) (1 + \gamma(t, y, \zeta)) d\nu(\zeta) dt \right. \\
& \left. \left. + \int_{\mathbb{R}} x(t, y) r(t, y, \zeta) c(t, y, \zeta) (1 + \gamma(t, y, \zeta)) \tilde{N}(dt, d\zeta) \right] \right] \quad (7.14) \quad \{\text{eq0.14}\}
\end{aligned}$$

Choose

$$\begin{aligned}
b(t, y) &:= -\beta(t, y) \\
c(t, \zeta) &:= -\frac{\gamma(t, y, \zeta)}{1 + \gamma(t, y, \zeta)} \quad (7.15) \quad \{\text{eq0.15}\}
\end{aligned}$$

Then (7.14) reduces to

$$\begin{aligned}
du(t, y) = & f(t, y) dt + k(t, y) x(t, y) q(t, y) dB(t) \\
& + \int_{\mathbb{R}} \{ x(t, y) r(t, y, \zeta) (1 + \gamma(t, y, \zeta)) [k(t, y) + k(t, y) c(t, y, \zeta)] \} \tilde{N}(dt, d\zeta), \quad (7.16) \quad \{\text{eq0.16}\}
\end{aligned}$$

where

$$\begin{aligned}
f(t, y) = & -k(t, y) E[\delta_Y(y) | \mathcal{F}_t] + u(t, y) \left[ \int_{\mathbb{R}} \gamma(t, y, \zeta) c(t, y, \zeta) d\nu(\zeta) + \beta(t, y) b(t, y) \right] \\
& + k(t, y) x(t, y) q(t, y) b(t, y) + k(t, y) \int_{\mathbb{R}} x(t, y) r(t, y, \zeta) c(t, y, \zeta) (1 + \gamma(t, y, \zeta)) d\nu(\zeta) \quad (7.17) \quad \{\text{eq0.17}\}
\end{aligned}$$

Now define

$$\begin{aligned}
v(t, y) &:= k(t, y) x(t, y) q(t, y) \\
w(t, y) &:= k(t, y) x(t, y) r(t, y, \zeta). \quad (7.18) \quad \{\text{eq0.18}\}
\end{aligned}$$

Then from (7.12) and (7.15) we get the following BSDE in the unknowns  $u, v, w$ :

$$\begin{aligned}
du(t, y) = & \left( -k(t, y) \mathbb{E}[\delta_Y(y) | \mathcal{F}_t] - u(t, y) \left[ \int_{\mathbb{R}} \frac{\gamma^2(t, y, \zeta)}{1 + \gamma(t, y, \zeta)} d\nu(\zeta) + \beta^2(t, y) \right] \right. \\
& \left. - \beta(t, y) v(t, y) - \int_{\mathbb{R}} \gamma(t, y, \zeta) w(t, y, \zeta) d\nu(\zeta) \right) dt \\
& + v(t, y) dB(t) + \int_{\mathbb{R}} w(t, y, \zeta) \tilde{N}(dt, d\zeta); \quad 0 \leq t \leq T \\
u(T, y) = & \theta k(T, y) \mathbb{E}[\delta_Y(y) | \mathcal{F}_T] \quad (7.19) \quad \{\text{eq0.19}\}
\end{aligned}$$

This is a linear BSDE which has a unique solution  $u(t, y) = p(t, y)x(t, y)k(t, y), v(t, y), w(t, y, \zeta)$ . In particular, we may regard

$$p(t, y)x(t, y) = \frac{u(t, y)}{k(t, y)} \quad (7.20) \quad \{\text{eq0.20}\}$$

as known.

Maximizing  $\int_{\mathbb{R}} H dy_2$  with respect to  $c$  gives the first order equation

$$\int_{\mathbb{R}} \left\{ \frac{1}{c(t, y_1)} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] - x(t, y)p(t, y) \right\} dy_2 = 0, \quad (7.21)$$

i.e.,

$$c(t, y_1) = \hat{c}(t, y_1) = \frac{\int_{\mathbb{R}} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] dy_2}{\int_{\mathbb{R}} x(t, y)p(t, y) dy_2}. \quad (7.22) \quad \{\text{eq0.22}\}$$

Minimizing  $\int_{\mathbb{R}} H dy_1$  with respect to  $\mu$  gives the first order equation

$$\int_{\mathbb{R}} \{ \mu(t, y_2) \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] + x(t, y)p(t, y) \} dy_1 = 0, \quad (7.23)$$

i.e.

$$\mu(t, y_2) = \hat{\mu}(t, y_2) = \frac{\int_{\mathbb{R}} x(t, y)p(t, y) dy_1}{\int_{\mathbb{R}} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] dy_1}. \quad (7.24) \quad \{\text{eq0.24}\}$$

We can now verify that  $\hat{c}, \hat{\mu}$  satisfies all the conditions of the sufficient maximum principle, and hence we conclude the following:

**Theorem 7.1 (Optimal consumption for an insider under model uncertainty)** *The solution  $(c^*, \mu^*)$  of the stochastic differential game (7.2) is given by*

$$c^*(t, Y_1) = \frac{\int_{\mathbb{R}} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] dy_2|_{y_1=Y_1}}{\int_{\mathbb{R}} x(t, y)p(t, y) dy_2|_{y_1=Y_1}}. \quad (7.25) \quad \{\text{eq0.25}\}$$

and

$$\mu^*(t, Y_2) = \frac{\int_{\mathbb{R}} x(t, y)p(t, y) dy_1|_{y_2=Y_2}}{\int_{\mathbb{R}} \mathbb{E}[\delta_Y(y)|\mathcal{F}_t] dy_1|_{y_2=Y_2}}, \quad (7.26) \quad \{\text{eq0.26}\}$$

where  $h(t, y) = x(t, y)p(t, y)$  is given by (7.19)-(7.20).

## 7.2 Optimal insider portfolio under model uncertainty

Consider a financial market with two investment possibilities:

- (i) A risk free investment possibility with unit price  $S_0(t) = 1$  for all  $t \in [0, T]$
- (ii) A risky investment, where the unit price  $S(t) = S(t, Y)$  is modelled by the (forward) SDE

$$dS(t, Y) = S(t, Y)[(\alpha(t, Y) + \mu(t))dt + \beta(t, Y)dB(t)]; S(0) > 0. \quad (7.27) \quad \{\text{eq00.1}\}$$

Here  $\alpha(t, Y), \beta(t, Y)$  are given  $\mathbb{H}$ -adapted coefficients, while  $\mu(t)$  is a perturbation of the drift term, representing the model uncertainty chosen by the environment (player number 2).

Suppose the wealth process  $X(t, Y) = X^{\pi, \mu}(t, Y)$  associated to an insider portfolio  $\pi(t, Y)$  (representing the fraction of the wealth invested in the risky asset) is given by:

$$\begin{cases} dX(t, Y) = \pi(t, Y)X(t, Y)[(\alpha(t, Y) + \mu(t))dt + \beta(t, Y)dB(t)] \\ X(0) = x > 0 \end{cases} \quad (7.28) \quad \{\text{eq00.2}\}$$

Define the performance functional by

$$J(\pi, \mu) = \mathbb{E}\left[\int_0^T \frac{1}{2}\mu^2(t)dt + \theta(T, Y) \log X(T)\right], \quad (7.29) \quad \{\text{eq00.3}\}$$

where  $\theta(T, Y) > 0$  is a given  $\mathcal{H}_T$ -measurable random variable, and  $\frac{1}{2}\mu^2(t)$  represents a penalty rate, penalizing  $\mu$  for being away from 0. We assume that  $\pi$  is  $\mathbb{H}$ -adapted, while  $\mu$  is  $\mathbb{F}$ -adapted, i.e. has no inside information.

We want to find  $\pi^* \in \mathcal{A}_1$  and  $\mu^* \in \mathcal{A}_2$  such that

$$\sup_{\pi \in \mathcal{A}_1} \inf_{\mu \in \mathcal{A}_2} J(\pi, \mu) = \inf_{\mu \in \mathcal{A}_2} \sup_{\pi \in \mathcal{A}_1} J(\pi, \mu) = J(\pi^*, \mu^*). \quad (7.30) \quad \{\text{eq00.4}\}$$

We rewrite this problem as a classical stochastic differential game with one parameter  $y_1 = y \in \mathbb{R}$ . Thus we define

$$\begin{cases} dx(t, y) = \pi(t, y)x(t, y)[\{\alpha(t, y) + \mu(t)\}dt + \beta(t, y)dB(t)] \\ x(0, y) = x(y) > 0 \end{cases} \quad (7.31) \quad \{\text{eq00.5}\}$$

and

$$J(\pi(\cdot, y), \mu(\cdot)) = \mathbb{E}\left[\int_0^T \frac{1}{2}\mu^2(t)\mathbb{E}[\delta_Y(y)|\mathcal{F}_t]dt + \theta \log x(T, y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]\right]. \quad (7.32) \quad \{\text{eq00.6}\}$$

The Hamiltonian for this problem is

$$H(t, x, y, \pi, \mu, p, q) = \frac{1}{2}\mu^2\mathbb{E}[\delta_Y(y)|\mathcal{F}_t] + \pi x(\alpha(t, y) + \mu)p + \pi x\beta(t, y)q \quad (7.33) \quad \{\text{eq00.7}\}$$



and the BSDE for the adjoint processes  $p, q$  is

$$\begin{cases} dp(t, y) = -[\pi(t, y)\{(\alpha(t, y) + \mu(t))p(t, y) + \beta(t, y)q(t, y)\}]dt + q(t, y)dB(t) \\ p(T, y) = \frac{\theta(T, y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]}{x(T, y)}. \end{cases} \quad 0 \leq t \leq T \quad (7.34) \quad \{\text{eq00.8}\}$$

Maximizing  $H$  with respect to  $\pi$  gives the first order equation

$$x(t, y)[(\alpha(t, y) + \mu(t))p(t, y) + \beta(t, y)q(t, y)] = 0. \quad (7.35)$$

Since  $x(t, y) > 0$  and  $\beta(t, y) \neq 0$ , we deduce that

$$(\alpha(t, y) + \mu(t))p(t, y) + \beta(t, y)q(t, y) = 0 \quad (7.36)$$

and

$$q(t, y) = -\frac{\alpha(t, y) + \mu(t)}{\beta(t, y)}p(t, y). \quad (7.37)$$

Hence (7.32) reduces to

$$\begin{cases} dp(t, y) = -\frac{\alpha(t, y) + \mu(t)}{\beta(t, y)}p(t, y)dB(t) \\ p(T, y) = \frac{\theta(T, y)\mathbb{E}[\delta_Y(y)|\mathcal{F}_T]}{x(T, y)}. \end{cases} \quad (7.38) \quad \{\text{eq00.12}\}$$

Define

$$h(t, y) = p(t, y)x(t, y). \quad (7.39) \quad \{\text{eq00.13}\}$$

Then by the Itô formula we get

$$\begin{cases} dh(t, y) &= (\pi(t, y)\beta(t, y) - \frac{\alpha(t, y) + \mu(t)}{\beta(t, y)})h(t, y)dB(t) \\ h(T, y) &= p(T, y)x(T, y) = \theta E[\delta_Y(y)|\mathcal{F}_T]. \end{cases} \quad (7.40) \quad \{\text{eq00.14}\}$$

This BSDE has the solution

$$h(t, y) = \theta E[\delta_Y(y)|\mathcal{F}_t]. \quad (7.41) \quad \{\text{eq00.15}\}$$

Moreover, by the generalized Clark-Ocone formula we have

$$(\pi(t, y)\beta(t, y) - \frac{\alpha(t, y) + \mu(t)}{\beta(t, y)})h(t, y) = D_t h(t, y) = \theta E[D_t \delta_Y(y)|\mathcal{F}_t], \quad (7.42) \quad \{\text{eq00.16}\}$$

from which we get the following expression for our candidate  $\hat{\pi}(t, y)$  for the optimal portfolio

$$\hat{\pi}(t, y) = \frac{\alpha(t, y) + \hat{\mu}(t)}{\beta^2(t, y)} + \frac{E[D_t \delta_Y(y)|\mathcal{F}_t]}{\beta(t, y)E[\delta_Y(y)|\mathcal{F}_t]}, \quad (7.43) \quad \{\text{eq00.17}\}$$

where  $\hat{\mu}(t)$  is the corresponding candidate for the optimal perturbation.

Minimizing  $\int_{\mathbb{R}} H dy$  with respect to  $\mu$  gives the following first order equation for the optimal  $\hat{\mu}(t)$ :

$$\int_{\mathbb{R}} \{\hat{\mu}(t) \mathbb{E}[\delta_Y(y) | \mathcal{F}_t] + \hat{\pi}(t, y) \hat{x}(t, y) \hat{p}(t, y)\} dy = 0, \quad (7.44)$$

i.e.,

$$\hat{\mu}(t) = - \frac{\int_{\mathbb{R}} \hat{\pi}(t, y) \hat{x}(t, y) \hat{p}(t, y) dy}{\int_{\mathbb{R}} \hat{x}(t, y) \hat{p}(t, y) dy} = - \int_{\mathbb{R}} \hat{\pi}(t, y) E[\delta_Y(y) | \mathcal{F}_t] dy. \quad (7.45) \quad \{\text{eq00.19}\}$$

We can now verify that  $(\hat{\pi}, \hat{\mu})$  satisfies all the conditions of the sufficient maximum principle, and hence we conclude the following:

**Theorem 7.2 (Optimal portfolio for an insider under model uncertainty)** *The saddle point  $(\pi^*(t, Y), \mu^*(t))$ , where  $\pi^*(t, Y) = \pi^*(t, y)|_{y=Y}$ , of the stochastic differential game (7.28) is given by the solution of the following coupled system of equations*

$$\pi^*(t, y) = \frac{\alpha(t, y) + \mu^*(t)}{\beta^2(t, y)} + \frac{E[D_t \delta_Y(y) | \mathcal{F}_t]}{\beta(t, y) E[\delta_Y(y) | \mathcal{F}_t]}, \quad (7.46) \quad \{\text{eq00.20}\}$$

and

$$\mu^*(t) = - \int_{\mathbb{R}} \pi^*(t, y) E[\delta_Y(y) | \mathcal{F}_t] dy. \quad (7.47) \quad \{\text{eq00.21}\}$$

*Remark 7.3 This result is an extension to insider trading of a result in [OS4].*

Consider the special case when  $Y$  is a Gaussian random variable of the form

$$Y = Y(T_0); \text{ where } Y(t) = \int_0^t \psi(s) dB(s), \text{ for } t \in [0, T_0] \quad (7.48) \quad \{\text{eqY}\}$$

for some deterministic function  $\beta \in \mathbf{L}^2[0, T_0]$  with

$$\|\psi\|_{[0, T]}^2 := \int_0^T \psi(s)^2 ds > 0 \text{ for all } t \in [0, T]. \quad (7.49)$$

In this case it is well known that the Donsker delta functional is given by

$$\delta_Y(y) = (2\pi v)^{-\frac{1}{2}} \exp^{\circ} \left[ -\frac{(Y - y)^{\circ 2}}{2v} \right] \quad (7.50)$$

where we have put  $v := \|\psi\|_{[0, T_0]}^2$ . See e.g. [AaOU], Proposition 3.2. We have

$$\mathbb{E}[\delta_Y(y) | \mathcal{F}_t] = (2\pi \|\psi\|_{[t, T_0]}^2)^{-\frac{1}{2}} \exp \left[ -\frac{(Y(t) - y)^2}{2\|\psi\|_{[t, T_0]}^2} \right]. \quad (7.51)$$

and

$$\mathbb{E}[D_t \delta_Y(y) | \mathcal{F}_t] = -(2\pi \|\psi\|_{[t, T_0]}^2)^{-\frac{1}{2}} \exp \left[ -\frac{(Y(t) - y)^2}{2\|\psi\|_{[t, T_0]}^2} \right] \frac{Y(t) - y}{\|\psi\|_{[t, T_0]}^2} \psi(t). \quad (7.52)$$

For more details see [DrØ].

**Corollary 7.4** *Suppose that  $Y$  is Gaussian of the form (7.48). Then the saddle point  $(\pi^*(t, Y), \mu^*(t))$ , where  $\pi^*(t, Y) = \pi^*(t, y)|_{y=Y}$ , of the stochastic differential game (7.28) is given by the solution of the following coupled system of equations*

$$\pi^*(t, Y) = \frac{\alpha(t, y) + \mu^*(t)}{\beta^2(t, y)} + \frac{Y(T_0) - Y(t)}{\beta(t, y)\|\psi\|_{[t, T_0]}^2} \psi(t), \quad (7.53) \quad \{\text{eq00.20}\}$$

and

$$\mu^*(t) = -\theta(2\pi\|\psi\|_{[t, T_0]}^2)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\left[-\frac{(Y(t) - y)^2}{2\|\psi\|_{[t, T_0]}^2}\right] \pi^*(t, y) dy. \quad (7.54) \quad \{\text{eq00.21}\}$$

**Corollary 7.5** *Suppose that  $Y = B(T_0)$  for some  $T_0 > T$ . Then the saddle point  $(\pi^*(t, Y), \mu^*(t))$ , where  $\pi^*(t, Y) = \pi^*(t, y)|_{y=Y}$ , of the stochastic differential game (7.28) is given by the solution of the following coupled system of equations*

$$\pi^*(t, y) = \frac{\alpha(t, y) + \mu^*(t)}{\beta^2(t, y)} + \frac{B(T_0) - B(t)}{\beta(t, y)(T_0 - t)}; \quad 0 \leq t \leq T \quad (7.55) \quad \{\text{eq00.20}\}$$

and

$$\mu^*(t) = -\theta(2\pi(T_0 - t))^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\left[-\frac{(B(t) - y)^2}{2(T_0 - t)}\right] \pi^*(t, y) dy; \quad 0 \leq t \leq T. \quad (7.56) \quad \{\text{eq00.21}\}$$

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